Logic and Incidence Geometry

February 27, 2013

1 Informal Logic

Logic Rule 0. No unstated assumption may be used in a proof.

2 Theorems and Proofs

If [hypothesis] *then* [conclusion].

LOGIC RULE 1. The following are the six types of justifications allowed for statements in proofs:
(1) “By hypothesis ...”
(2) “By axiom ...”
(3) “By theorem ...” (previously proved)
(4) “By definition ...”
(5) “By step ...” a previous step in the argument)
(6) “By rule ... of logic.”

3 Reductio ad Absurtum (RAA)

Reductio ad absurtum is reduction to absurdity.

LOGIC RULE 2. To prove a statement $H \Rightarrow C$, assume the negation of the statement C (RAA hypothesis) and deduce an absurd statement using the hypothesis H if needed in your deduction.

Example. If l and m are distinct lines and are not parallel, the l and m have a unique point in common.

Proof. Since l and m are not parallel, they meet in a common point A. To show the uniqueness, suppose they meet in another point B other than A. Then l and m are two distinct lines passing through both A and B. This is contradictory to that two points determine a unique line.

4 Negation

If S is a statement, we denote the negation or contrary of S by $\neg S$, meaning “not S.”

LOGIC RULE 3. The statement “$\neg(\neg S)$” is the same as “S.”

LOGIC RULE 4. The statement “$\neg(H \Rightarrow C)$” is the same as “$H \wedge \neg C$.”

LOGIC RULE 5. The statement “$\neg(S_1 \wedge S_2)$” is the same as “$\neg S_1 \vee \neg S_2$.”

5 Quantifiers

Universal quantifier:

"For any x, ..."
"For each x, ..."
"For every x, ..."
"For all x, ..."
"Given any x, ..."

Existential quantifiers:
"For some x, ..."
"There exists an x, ..."
"There is an x, ..."
"There are x, ..."
"There exists a unique x, ..."

LOGIC RULE 6. The statement "$\neg(\forall x\ S(x))$" is the same as "$\exists x\ \neg S(x)$."

LOGIC RULE 7. The statement "$\neg(\exists x\ S(x))$" is the same as "$\forall x\ \neg S(x)$."

6 Implication

Rule of detachment or *modus ponens*:

LOGIC RULE 8. If $P \Rightarrow Q$ and P are steps in a proof, then Q is a justifiable step.

LOGIC RULE 9. (a) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$.
(b) $(P \wedge Q) \Rightarrow P$; $(P \wedge Q) \Rightarrow Q$.
(c) "$P \Leftrightarrow Q$" means that "$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$."

7 Law of Excluded Middle and Proof by Cases

LOGIC RULE 10. For every statement P, "$P \vee \neg P$" is a valid step in a proof (law of excluded middle).

LOGIC RULE 11. Suppose the disjunction of statements $S_1, \ldots, S_n$ is already a valid step in a proof. Suppose that proofs of C are carried out from each of the case assumptions $S_1, \ldots, S_n$. Then C can be concluded as a valid step in the proof. That is,

$$[(S_1 \vee \cdots \vee S_n) \wedge (S_1 \Rightarrow C) \wedge \cdots \wedge (S_n \Rightarrow C)] \Rightarrow C.$$

8 Incidence Geometry

Incidence Axiom 1 (IA1). For every pair of points P and Q there exists a unique line l incident with P and Q.

Incidence Axiom 2 (IA2). For every line l there exist at least two distinct points incident with l.

Incidence Axiom 3 (IA3). There exist three distinct points A, B, C not simultaneously incident with a common line l.

Proposition 8.1. *If l, m are distinct lines and are not parallel, then l and m meet at a unique point.*

Proof. The lines l, m must intersect since they are not parallel. Suppose they meet at two distinct points P, Q. Then there are two lines l, m through points P, Q. This is contradictory to Incidence Axiom 1. So l and m meet at a unique point. $\square$

Proposition 8.2. *There exist three distinct lines not through any common point.*

Proof. Let A, B, C be three distinct points not collinear. Since any two distinct points determine a unique line, the three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ must be distinct. We claim that the three lines have no common point. Suppose they have a common point P. We must have $P \neq A$. (Otherwise, $P = A$ implies that A lies on the line $\overleftrightarrow{BC}$ since P is a common point of $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$. Thus A, B, C are collinear, a contradiction.) Then two lines $\overleftrightarrow{AB}, \overleftrightarrow{AC}$ pass through points P, A. This is contradictory to Incidence Axiom 1. $\square$

Proposition 8.3. *For every line there is at least one point not lying on it.*

Proof. Let A, B, C be three distinct points not collinear by Incidence Axiom 3. Suppose there is a line l which has no point outside l, i.e., l contains every point. Then l contains all A, B, C. This means that A, B, C are collinear, a contradction. $\square$

Proposition 8.4. *For every point there is at least one line not passing through it.*

Proof. Let l, m, n be three distinct lines not concurrent by Proposition 8.2. Given an arbitrary point P; then one of l, m, n does not pass through P. $\square$

Proposition 8.5. *For every point P there exist at least two lines through P.*

Proof. Let A, B, C be three points not collinear. If P is outside the line $\overleftrightarrow{AB}$, then the lines $\overleftrightarrow{AP}, \overleftrightarrow{BP}$ pass through point P and must be distinct. Otherwise, $\overleftrightarrow{AP}=\overleftrightarrow{BP}$ implies that A, B, P are collinear; so P is on the line $\overleftrightarrow{AB}$, contradicting to that P is outside $\overleftrightarrow{AB}$. If P is on the line $\overleftrightarrow{AB}$, then $\overleftrightarrow{AB}, \overleftrightarrow{CP}$ are two distinct lines passing through P. $\square$

9 Models

We may use dots and dashes to to represent points and lines so that the axioms appear to be correct statements. We view these dots and dashes as a **model** for the incidence geometry.

Example 1. Consider a set $\{A, B, C\}$ of three letters, which are called "points." The "lines" are those subsets consisting of two letters, i.e., $\{A, B\}$, $\{A, C\}$, $\{B, C\}$. A "point" is interpreted as "incident" with a "line" if it is a member of that subset. For instance, point A is lies on lines $\{A, B\}$ and $\{A, C\}$. Every two distinct lines meet at a unique point (referred to **elliptic parallel property**). There are no parallel lines. **It is impossible in incidence geometry to prove that parallel lines exist.**

Example 2. Let S^2 denote a sphere. Consider each member of S^2 as a "point" and each great circle of S^2 as a "line." Then there are no parallel lines. However, the "points" and "lines" do not form an incidence geometry, since Axiom 1 is not satisfied.

Example 3 (The smallest affine plane). Consider a set $\{A, B, C, D\}$ of four letters, called "points." The "lines" are those subsets consisting of two letters. Then $\{A, D\}$ and $\{B, C\}$ are parallel lines. The elliptic parallel property cannot be proved. (**It is impossible in incidence geometry to prove that parallel lines do not exist.**) For the line $\{A, B\}$ and the point C outside $\{A, B\}$, there is a unique line $\{C, D\}$ passing through C

and parallel to $\{A, B\}$. The incidence geometry satisfies the Euclidean parallel property. So the four points and six lines form an affine plane.

Example 4. Consider a set $\{A, B, C, D, E\}$ of five letters, called "points." The "lines" are those subsets consisting of two letters. There are exactly 10 lines. Let "incidence" be set membership. The elliptic parallel property is not satisfied. For the line $\{A, B\}$ and the point C, there are two lines $\{C, D\}, \{C, E\}$ passing through C and parallel to the line $\{A, B\}$.

Definition 1 (Independence)**.** A statement is said to be **independent** if it is impossible to be either proved or disproved.

Definition 2 (Completeness)**.** An axiom system is said to be **complete** it there is no independent statement in the language of the system, i.e., every statement in the language of the system can be either proved or disproved from the axioms.

The axioms of incidence geometry is *not* complete. The axioms of Euclidean and hyperbolic geometries are complete (see Tarski's article in Henkin, Suppes, and Tarski, 1959).

10 Isomorphism of Models

Two models of incidence geometry are said to be **isomorphic** if there is a one-to-one correspondence $P \leftrightarrow P'$ between the points of the models and a one-to-one correspondence $\ell \leftrightarrow \ell'$ between the lines of the models such that P lies on ℓ if and only if P' lies on ℓ'. Such a correspondence is called an **isomorphism** from one model onto the other.

Example 5. Consider a set $\{a, b, c\}$ of three letters, which we will call "lines" now. The "points" will be those subsets that contain exactly two letters — $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$. Let "incidence" be the set membership. For instance, "point" $\{a, b\}$ is on the "lines" a and b, but not on the "line" c. This model seems to be structurally the same as the three-point model in Example 1, except the change of notations. An explicit isomorphism is given as follows:

$$\begin{gathered} A \leftrightarrow \{a, b\}, \quad B \leftrightarrow \{b, c\}, \quad C \leftrightarrow \{a, c\}, \\ \{A, B\} \leftrightarrow b, \quad \{B, C\} \leftrightarrow c, \quad \{A, C\} \leftrightarrow a. \end{gathered}$$

For instance, the point A lies on the two lines $\{A, B\}$ and $\{A, C\}$ only; its corresponding point $\{a, b\}$ lies on the corresponding two lines b and a only. The line $\{A, B\}$ is incident with the two points A and B only; the corresponding line b is incident with the corresponding two points $\{a, b\}$ and $\{b, c\}$ only.

The point $B \in \{A, B\}, B \in \{B, C\}$ correspond to $\{b, c\} \ni b, \{b, c\} \ni c$. The point $C \in \{A, C\}, C \in \{B, C\}$ correspond to $\{a, c\} \ni a, \{a, c\} \ni c$. So the bijection is an isomorphism. However, the following one-to-one correspondence

$$\begin{gathered} A \leftrightarrow \{a, b\}, \quad B \leftrightarrow \{b, c\}, \quad C \leftrightarrow \{a, c\}, \\ \{A, B\} \leftrightarrow a, \quad \{B, C\} \leftrightarrow b, \quad \{A, C\} \leftrightarrow c. \end{gathered}$$

is not an isomorphism, since the point A lies in the line $\{A, C\}$, but the corresponding point $\{a, b\}$ does not lie on the corresponding line c.

Points: `chair`, `table`, `board`. Lines: `Classroom`, `Canteen`, `Hall`. Incidence Relation: `chair` $\sim$ `Classroom`, `table` $\sim$ `Classroom`, `chair` $\sim$ `Canteen`, `board` $\sim$ `Canteen`, `table` $\sim$ `Hall`, `board` $\sim$ `Hall`. Then the points and lines and the incidence relation form an incidence geometry. This incidence geometry is isomorphic to the incidence geometry in Example 1.

11 Affine and Projective Planes

Definition of Affine Plane. A model of incidence geometry having the Euclidean parallel property.

Definition of Projective Plane. A model of incidence geometry satisfying the **Elliptic parallel property** (any two lines meet) and that every line has at least three points.

Example 6 (The smallest projective plane). Consider the set of seven points A, B, C, D, E, F, G, and the set of seven lines lines $\{A, B, F\}$, $\{A, C, E\}$, $\{A, D, G\}$, $\{B, C, D\}$, $\{B, E, G\}$, $\{C, F, G\}$, $\{D, E, F\}$. Then the seven points and the seven lines form a projective plane which can be demonstrated by Figure 1.

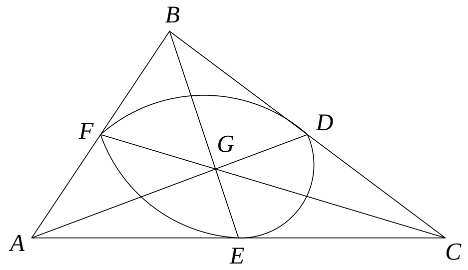

Figure 1: Smallest projective plane

Model of Projective Plane. Let $\mathscr{A}$ be an affine plane. We introduce an equivalence relation $\sim$ on the lines of $\mathscr{A}$:

$$l \sim m \quad \text{if either “}l = m\text{” or “}l \| m.\text{”}$$

(Indeed, $\sim$ is obviously reflexive and symmetric. Assume $l \sim m$ and $m \sim n$. Suppose $l \not\sim n$. Then l, m, n are distinct, $l \| m$, $n \| m$, and l, n meet at a point P. Then there two distinct lines l, n parallel to m and through P. This is contradictory to Euclidean parallel property. So we must have $l \sim n$. Thus $\sim$ is transitive.) We denote by $[l]$ the set of all lines parallel to a line l, called the **equivalence class** of l. Each equivalence class of parallel lines is called a **point at infinity**.

We now enlarge the model $\mathscr{A}$ to a new model $\mathscr{A}^*$ by adding these points at infinity; the points of $\mathscr{A}$ are called **ordinary** points of $\mathcal{A}^*$ for emphasis. We further enlarge the incidence relation by specifying that each point at infinity (an equivalence class of parallel lines) lies on every line in that equivalence class: $[l]$ lies on l and on every line m such that $l \| m$. We add one new line l_∞, the **line at infinity**, consisting of all points at infinity in $\mathscr{A}^*$. The line l_∞ is incident with every its point. Then $\mathscr{A}^*$ becomes a projective plane, called the **projective completion** of $\mathscr{A}$.

Verification of IA1. Let P and Q be two distinct points of $\mathscr{A}^*$. If P, Q are both ordinary points, they lie on a unique line l in $\mathscr{A}$. Clearly, the line l_∞ does not pass through either of P, Q. So l is the unique line in $\mathscr{A}^*$ incident with both P, Q.

If one of them is a point at infinity, say, P is a point at infinity $[l]$ and Q is an ordinary point, then either Q is on l or Q is not on l. In the formal case, we have P, Q incident with l. The line l_∞ does not pass through both P, Q. If m is a line in $\mathscr{A}$ through both P, Q, then $m \in [l] = P$, i.e., $l \| m$ and Q is on m; it forces $l = m$. In the latter case, since Q is not on l, there exists a unique line m through Q and parallel to l. Then $[m] = [l] = P$. Thus l is the unique line in $\mathscr{A}^*$ passing through both P, Q.

If both P, Q are two distinct points at infinity, then the line l_∞ is the unique line passing through P, Q.

Verification of IA2. Each line l in $\mathscr{A}$ already has two points of $\mathscr{A}$ on m. Adding the point $[l]$ at infinity, the line l has at least three points in $\mathscr{A}^*$. Let A, B, C be distinct points not collinear by IA3. Then no two of the three lines $\overleftrightarrow{AB}, \overleftrightarrow{AC}, \overleftrightarrow{BC}$ are parallel. Then $[\overleftrightarrow{AB}], [\overleftrightarrow{AC}], [\overleftrightarrow{BC}]$ are three points on l_∞.

Verification of IA3. It already hold in $\mathscr{A}$.

Verification of Elliptic Parallel Property. For ordinary two distinct lines l, m in $\mathscr{A}$, if they do not meet in $\mathscr{A}$, i.e., they are parallel in $\mathscr{A}$, then they meet at the point $[l]$ $(= [m])$ in $\mathscr{A}^*$. For an ordinary line l in $\mathscr{A}$ and l_∞, they meet at the point $[l]$ in $\mathscr{A}^*$.

Example 7 (A model of real projective plane). Let $\mathbf{S}$ be a sphere centered at the origin O of our 3-dimension Euclidean space, and $\mathbf{T}$ the tangent plane of $\mathbf{S}$ at the north pole N. Put a candle at O, then the upper (open) hemisphere is projected to the tangent plane $\mathbf{T}$ in one-to-one correspondence. Each line in $\mathbf{T}$ is the image of a great arc in the upper open hemisphere. We introduce points at infinity on $\mathbf{T}$ by considering the upper closed hemisphere, where each pair of antipodal points on the equator are identified to one point. The points and great arcs in the closed upper hemisphere with the identification form a model of real projective plane. Each point $[l]$ at infinity (consisting of all lines m parallel to l) can be represent by an identification of two antipodal points of the closed great arcs corresponding to l. The the points and lines (=great arcs in the upper hemisphere with antipodal points identified) become a model of real projective plane.

Example 8 (Another model of real projective plane). Let $\mathbb{R}^3$ be the vector space of all tuples (x, y, z), where $x, y, z \in \mathbb{R}$. Define an equivalence relation $\sim$ on the set $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$:

$$(x, y, , z) \sim (x', y', z') \;\Leftrightarrow\; (x', y', z') = k(x, y, z) \text{ for a nonzero } k \in \mathbb{R}.$$

Let $\mathbf{P}$ denote the set of all "points" $[x, y, z]$, the equivalence classes of elements (x, y, z) in $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$ under $\sim$. A "line" is a set of all points $[x, y, z]$ such that $ux + vy + wz = 0$ for a fixed tuple $(u, v, w) \in \mathbb{R}^3 \smallsetminus \{(0,0,0)\}$, denoted $l(u, v, w)$. A point $[x, y, z]$ is "incident" with a line l if $[x, y, z]$ is a member of l. Define incidence to be the membership relation. Then so defined the points, lines, and incidence form a projective geometry.

Analogously, take "points" to be straight lines of $\mathbb{R}^3$ through the origin, "lines" to be the plane of $\mathbb{R}^3$ through the origin, and incidence to be the membership relation. Then the points, lines, and the incidence form a projective geometry.

Verification of IA1: Two distinct points $[x, y, z], [x', y', z']$ determine a unique line $l(u, v, w)$, where $(u, v, w) := (x, y, z) \times (x', y', z')$.

Verification of IA2: Trivial that each lines contains at least three points.

Verification of IA3: Trivial.

Verification of Elliptic Parallel Property: The intersection of every two distinct 2-dimensional subspaces is a 1-dimensional subspace. So every two distinct lines meet at a point.

Dual Projective Plane. Let $\mathscr{G}$ be a projective plane. Let $\mathscr{G}'$ denote a geometry whose set of points is the set of lines in $\mathscr{G}$, and whose set of lines is the set of points in $\mathscr{G}$. The incidence relation in $\mathscr{G}'$ is the same incidence relation in $\mathscr{G}$. Then $\mathscr{G}'$ is a also a projective plane, called the **dual projective plane** of $\mathscr{G}$.

Verification of IA1: This is equivalent to verify that two distinct lines meet at a unique point in $\mathscr{G}$, which is Proposition 8.1.

Verification of IA2: Need to show that each line in $\mathscr{G}'$ has at least three points. This is equivalent to show that through each point in $\mathscr{G}$ there are at least three lines. In fact, given a point P, let l be a line not through P. Let A, B, C be three distinct points on l. Then $\overleftrightarrow{AP}, \overleftrightarrow{BP}, \overleftrightarrow{CP}$ are three distinct lines through P.

Verification of IA3: This is equivalent to verify that in $\mathscr{G}$ there are three distinct lines not through a common point, which is Proposition 8.2.

Verification of Elliptic Parallel Property: To show that in $\mathscr{G}'$ any two lines meet at a unique point, equivalently, any two points in $\mathscr{G}$ determines a unique line.

Principle of Duality in Projective Geometry. If there is a theorem about

points P_1, P_2, …, P_s, lines l_1, l_2, …, l_t, and their inductions,

then there is a corresponding theorem about

lines m_1, m_2, …, m_s, points Q_1, Q_2, …, Q_t, and their corresponding inductions.

Theorem 11.1 (**Desargues Theorem**). *Let A, B, C, A', B', C' be six distinct points in projective geometry such that the lines $\overleftrightarrow{AA'}, \overleftrightarrow{BB'}, \overleftrightarrow{CC'}$ are concurrent. Let lines $\overleftrightarrow{AB}, \overleftrightarrow{A'B'}$ meet at P, lines $\overleftrightarrow{AC}, \overleftrightarrow{A'C'}$ meet at Q, and lines $\overleftrightarrow{BC}, \overleftrightarrow{B'C'}$ meet at R. Then P, Q, R are collinear.*

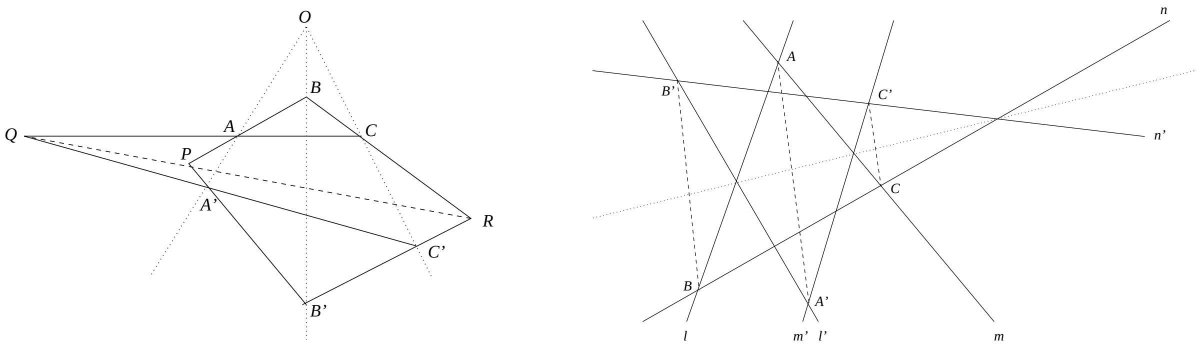

Figure 2: Desargues theorem and its dual

Theorem 11.2 (**Dual of Desargues Theorem**). *Let l, m, n, l', m', n' be six distinct lines in projective geometry such that $l \cap l', m \cap m', n \cap n'$ are collinear. Let p be the line through $l \cap m, l' \cap m'$; q the line through $l \cap n, l' \cap n'$; and r the line through $m \cap n, m' \cap n'$. Then p, q, r are concurrent.*

Projection. Let $\mathbf{P}$ denote the real projective plane. Let P be a point and l a line such that $P \notin l$. The **projection from** P **to** l is the mapping $\pi : \mathscr{P}(\mathbf{P}) \smallsetminus \{P\} \to l$ defined by

$$\pi(Q) = \overleftrightarrow{PQ} \cap l.$$

Using homogeneous coordinates $[x, y, z]$ to find a formula for the projection. Let $P = [x_0, y_0, z_0]$ and l be given by $ux + vy + wz = 0$, where $(u, v, w) \neq (0, 0, 0)$. Let $Q = [x_1, y_1, z_1]$.

If $u(x_1 - x_0) + v(y_1 - y_0) + w(z_1 - z_0) \neq 0$, then $\overleftrightarrow{PQ} \cap l$ is given by

$$\left[(x_0, y_0, z_0) - \frac{ux_0 + vy_0 + wz_0}{u(x_1 - x_0) + v(y_1 - y_0) + w(z_1 - z_0)}(x_1 - x_0, y_1 - y_0, z_1 - z_0)\right].$$

If $u(x_1 - x_0) + v(y_1 - y_0) + w(z_1 - z_0) = 0$, then $\overleftrightarrow{PQ} \cap l = [x_1 - x_0, y_1 - y_0, z_1 - z_0]$. Combined two together, we see that $\overleftrightarrow{PQ} \cap l$ is given by

$$[(ux_1 + vy_1 + wz_1)(x_0, y_0, z_0) - (ux_0 + vy_0 + wz_0)(x_1, y_1, z_1)]$$

This formula is true even when $P \in l$.